Hot Plasma Waves Surrounding the Schwarzschild Event Horizon in a Veselago Medium

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Abstract

This paper investigates wave properties of hot plasma in a Veselago medium. For the Schwarzschild black hole, the $3+1$ GRMHD equations are re-formulated which are linearly perturbed and then Fourier analyzed for rotating (non-magnetized and magnetized) plasmas. The graphs of wave vector, refractive index and change in refractive are used to discuss the wave properties. The results obtained confirm the presence of Veselago medium for both rotating (non-magnetized and magnetized) plasmas. This work generalized the isothermal plasma waves in the Veselago medium to hot plasma case.

1 Introduction

Plasmas are found nearly everywhere in nature. These are electrically conductive and give a strong respond to electromagnetic fields. Plasma fills the interplanetary and interstellar medium and are the components of the stars.

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To explore the dynamics of magnetized plasma and characteristics of black hole gravity when it acts in plasma's magnetic field, the theory of general relativistic magnetohydrodynamics (GRMHD) is the most accurate academic discipline. Schwarzschild black hole has zero angular momentum so plasma present in the magnetosphere moves only in the radial direction.

The gravity of black hole perturbs the magnetospheric plasma. Relativists are always curious to study the effect of these perturbations in the black hole regime. Regge and Wheeler [1] concluded that the Schwarzschild singularity remains stable when a small non-spherical odd-parity perturbation is introduced. Zerilli [2] explored the same stability problem by considering an even parity perturbation. The behavior of electric field generated by a charged particle at rest near the Schwarzschild black hole was discussed by Hanni and Ruffini [3]. Sakai and Kawata [4] examined electron-positron plasma waves in the frame of two fluid equations for the Schwarzschild black hole. Hirotoni and Tomimatsu [5] found that a small perturbation of polodial magnetic field could highly disturb the plasma accretion. They assumed non-stationary and axisymmetric perturbations of MHD accretion onto the Schwarzschild magnetosphere. Zenginoglu et al. [6] solved a hyperboloidal initial value problem for the Bardeen-Press equation to analyze the effect of gravitational perturbation on the Schwarzschild spacetime.

The $3 + 1$ formalism (also called Arnowitt, Deser and Misner (ADM) [7]) is much helpful to study gravitational radiations from black hole as well as analyzing the gravitational waves. To explore the magnificent aspects of general relativity (GR), many authors [8]-[10] adopted this technique. The electromagnetic theory in the black hole regime was developed by Thorne and Macdonald [11, 12]. Durrer and Straumann [13] deduced some basic results of GR by using this formalism. Holcomb and Tajima [14], Holcomb [15] and Dettmann et al. [16] studied the wave properties for the Friedmann universe. Buzzi et al. [17] investigated the plasma wave propagation close to the Schwarzschild magnetosphere. Zhang [18] composed the laws of perfect GRMHD in 3+1 formalism for a general spacetime. The same author [19] also described the role of cold plasma perturbation in the vicinity of the Kerr black hole. Using this formulation, Sharif and his collaborators [20]-[23] discussed plasma (cold, isothermal and hot) wave properties with non-rotating as well as rotating backgrounds.

Metamaterials are artificial materials that have unusual electromagnetic properties and Veselago medium or Double negative medium (DNG) is its most significant class. This medium has both electric permittivity as well

as magnetic permeability less than zero. It is also known as negative refractive index medium (NIM) and negative phase velocity medium (NPV). Many people [24]-[28] considered this medium to explore the unusual behavior of physical laws and their plausible applications. Ziolkowski [29] studied wave propagation in DNG medium both analytically and numerically. Valanju et al. [30] found positive and very inhomogeneous wave refraction in this unusual medium. Ramakrishna [31] examined the problem of designing such materials which have negative material parameters. He also discussed the concept of perfect lens consisting of a slab of negative refractive materials (NRM). Veselago [32] verified the concepts related to the energy, linear momentum and mass transferred by an electromagnetic wave in a negative refraction medium. In a recent paper [33], we have discussed the isothermal plasma wave properties for the Schwarzschild magnetosphere in a Veselago medium. The results verified the presence of this medium for only rotating non-magnetized plasma.

In this paper, we investigate wave properties of hot plasma in the vicinity of the Schwarzschild event horizon in a Veselago medium. The format of the paper is as follows. In Section **2**, the general line element in ADM $3 + 1$ formalism and its modification for the Schwarzschild planar analogue is given. Linear perturbation and Fourier analysis of the $3 + 1$ GRMHD equations for hot plasma is provided in section **3**. Sections **4** and **5** give the reduced form of the GRMHD equations for rotating (non-magnetized and magnetized respectively) plasmas. In the last section, summary of the results is given.

2 3+1 Foliation and Planar Analogue of the Schwarzschild Spacetime

The $3 + 1$ split of spacetime is an access to the field equations in which four-dimensional spacetime is sliced into three-dimensional spacelike hypersurfaces. In ADM $3 + 1$ formalism, the general line element is [19]

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \qquad (2.1)$$

where the ratio of the fiducial proper time to universal time, i.e., $\frac{d\tau}{dt}$ is denoted by α (lapse function). When FIDO (fiducial observer) changes his position from one hypersurface to another, the shift vector β^i calculates the change

of spatial coordinates. The components of three-dimensional hypersurfaces are denoted by γ_{ij} $(i, j = 1, 2, 3)$. A natural observer associated with the above spacetime is known as FIDO. The mathematical description of the Schwarzschild planar analogue is given by

$$ds^2 = -\alpha^2(z)dt^2 + dx^2 + dy^2 + dz^2, \qquad (2.2)$$

where the directions z, x and y are analogous to the Schwarzschild coordinates r, ϕ and θ respectively. The comparison of Eqs.(2.1) and (2.2) yields

$$\alpha = \alpha(z), \quad \beta = 0, \quad \gamma_{ij} = 1 \ (i = j). \qquad (2.3)$$

3   3+1 GRMHD Equations for the Schwarzschild Planar Analogue in a Veselago Medium

Appendix **A** contains the $3 + 1$ GRMHD equations in a Veselagho medium for the plasma existing in the general line element and the Schwarzschild planar analogue (Eqs.(2.1) and (2.2)). In the vicinity of the Schwarzschild magnetosphere, the specific enthalpy for hot plasma is [19]

$$\mu = \frac{\rho + p}{\rho_0}, \qquad (3.1)$$

where the rest mass density, moving mass density, pressure and specific enthalpy are represented by ρ_0, ρ, p and μ respectively. For cold plasma, we have $p = 0$ while for isothermal plasma, $p \neq 0$ but specific enthalpy is constant. However, specific enthalpy is variable for the hot plasma. This is the most general plasma which reduces to cold and isothermal plasmas with some restrictions. This equation shows the exchange of energy between the plasma and fluid's magnetic field. It is obvious from the above equation that for μ to be variable, p must be variable. In this paper, we have used the hot plasma along with the Veselago medium in the vicinity of the Schwarzschild magnetosphere. The $3 + 1$ GRMHD equations (Eqs.($A10$)-($A14$)) for hot plasma surrounding the Schwarzschild event horizon become

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\alpha \mathbf{V} \times \mathbf{B}), \qquad (3.2)$$

$$\nabla \cdot \mathbf{B} = 0, \qquad (3.3)$$

$$\frac{\partial(\rho + p)}{\partial t} + (\rho + p)\gamma^2 \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t} + (\rho + p)\gamma^2 V \cdot (\alpha \mathbf{V} \cdot \nabla)\mathbf{V}$$
$$+(\rho + p)\nabla \cdot (\alpha \mathbf{V}) = 0, \qquad (3.4)$$

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$$\left\{ \left((\rho + p)\gamma^2 + \frac{\mathbf{B}^2}{4\pi} \right) \delta_{ij} + (\rho + p)\gamma^4 V_i V_j - \frac{1}{4\pi} B_i B_j \right\} \left(\frac{1}{\alpha} \frac{\partial}{\partial t} \right.$$

$$+ \mathbf{V}.\nabla) V^j + \gamma^2 V_i (\mathbf{V}.\nabla)(\rho + p) - \left(\frac{\mathbf{B}^2}{4\pi} \delta_{ij} - \frac{1}{4\pi} B_i B_j \right) V_{,k}^j V^k$$

$$= -(\rho + p)\gamma^2 a_i - p_{,i} + \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B})_i \nabla.(\mathbf{V} \times \mathbf{B}) - \frac{1}{8\pi\alpha^2} (\alpha \mathbf{B})_{,i}^2$$

$$+ \frac{1}{4\pi\alpha} (\alpha B_i)_{,j} B^j - \frac{1}{4\pi\alpha} [\mathbf{B} \times \{\mathbf{V} \times (\nabla \times (\alpha \mathbf{V} \times \mathbf{B}))\}]_i, \qquad (3.5)$$

$$(\frac{1}{\alpha} \frac{\partial}{\partial t} + \mathbf{V}.\nabla)(\rho + p)\gamma^2 - \frac{1}{\alpha} \frac{\partial p}{\partial t} + 2(\rho + p)\gamma^2 (\mathbf{V}.\mathbf{a}) + (\rho + p)$$

$$\gamma^2 (\nabla.\mathbf{V}) - \frac{1}{4\pi\alpha} (\mathbf{V} \times \mathbf{B}).(\mathbf{V} \times \frac{\partial \mathbf{B}}{\partial t}) - \frac{1}{4\pi\alpha} (\mathbf{V} \times \mathbf{B}).(\mathbf{B} \times \frac{\partial \mathbf{B}}{\partial t})$$

$$+ \frac{1}{4\pi\alpha} (\mathbf{V} \times \mathbf{B}).(\nabla \times \alpha \mathbf{B}) = 0. \qquad (3.6)$$

In rotating background, plasma is assumed to flow in two dimensions, i.e., in xz-plane. Thus FIDO's measured magnetic field $\mathbf{B}$ and velocity $\mathbf{V}$ become

$$\mathbf{V} = V(z)\mathbf{e}_x + u(z)\mathbf{e}_z, \quad \mathbf{B} = B[\lambda(z)\mathbf{e}_x + \mathbf{e}_z], \qquad (3.7)$$

where B is an arbitrary constant. The quantities λ, u and V are related by [20]

$$V = \frac{V^F}{\alpha} + \lambda u, \qquad (3.8)$$

where V^F is an integration constant. Thus the Lorentz factor $\gamma = \frac{1}{\sqrt{1-\mathbf{V}^2}}$ takes the form

$$\gamma = \frac{1}{\sqrt{1 - u^2 - V^2}}. \qquad (3.9)$$

When the plasma flow is perturbed, the flow variables (mass density ρ, pressure p, velocity $\mathbf{V}$ and magnetic field $\mathbf{B}$) turn out to be

$$\rho = \rho^0 + \delta\rho = \rho^0 + \rho\widetilde{\rho}, \quad p = p^0 + \delta p = p^0 + p\widetilde{p},$$
$$\mathbf{V} = \mathbf{V}^0 + \delta\mathbf{V} = \mathbf{V}^0 + \mathbf{v}, \quad \mathbf{B} = \mathbf{B}^0 + \delta\mathbf{B} = \mathbf{B}^0 + B\mathbf{b}, \qquad (3.10)$$

where unperturbed quantities are denoted by ρ^0, p, $\mathbf{V}^0$, $\mathbf{B}^0$ while $\delta\rho$, δp, $\delta\mathbf{V}$, $\delta\mathbf{B}$ represent linearly perturbed quantities. The following dimensionless

quantities $\widetilde{\rho}$, $\widetilde{p}$, v_x, v_z, b_x and b_z are introduced for the perturbed quantities

$$\tilde{\rho} = \tilde{\rho}(t,z), \quad \tilde{p} = \tilde{p}(t,z), \quad \mathbf{v} = \delta\mathbf{V} = v_x(t,z)\mathbf{e}_x + v_z(t,z)\mathbf{e}_z,$$

$$\mathbf{b} = \frac{\delta\mathbf{B}}{B} = b_x(t,z)\mathbf{e}_x + b_z(t,z)\mathbf{e}_z. \tag{3.11}$$

When we insert these linear perturbations in the perfect GRMHD equations (Eqs.(3.2)-(3.6)), it follows that

$$\frac{\partial(\delta\mathbf{B})}{\partial t} = -\nabla \times (\alpha\mathbf{v} \times \mathbf{B}) - \nabla \times (\alpha\mathbf{V} \times \delta\mathbf{B}), \tag{3.12}$$

$$\nabla.(\delta\mathbf{B}) = 0, \tag{3.13}$$

$$\frac{1}{\alpha}\frac{\partial(\delta\rho+\delta p)}{\partial t} + (\rho+p)\gamma^2\mathbf{V}.(\frac{1}{\alpha}\frac{\partial}{\partial t} + \mathbf{V}.\nabla)\mathbf{v} + (\rho+p)(\nabla.\mathbf{v})$$
$$= -2(\rho+p)\gamma^2(\mathbf{V}.\mathbf{v})(\mathbf{V}.\nabla)\ln\gamma - (\rho+p)\gamma^2(\mathbf{V}.\nabla\mathbf{V}).\mathbf{v}$$
$$+(\rho+p)(\mathbf{v}.\nabla\ln u), \tag{3.14}$$

$$\left\{ \left((\rho+p)\gamma^2 + \frac{\mathbf{B}^2}{4\pi} \right)\delta_{ij} + (\rho+p)\gamma^4 V_i V_j - \frac{1}{4\pi}B_i B_j \right\}\frac{1}{\alpha}\frac{\partial v^j}{\partial t}$$

$$+\frac{1}{4\pi}[\mathbf{B} \times \{\mathbf{V} \times \frac{1}{\alpha}\frac{\partial(\delta\mathbf{B})}{\partial t}\}]_i + (\rho+p)\gamma^2 v_{i,j}V^j + (\rho+p)$$
$$\times\gamma^4 V_i v_{j,k}V^j V^k + \gamma^2 V_i(\mathbf{V}.\nabla)(\delta\rho+\delta p) + \gamma^2 V_i(\mathbf{v}.\nabla)(\rho+p)$$
$$+\gamma^2 v_i(\mathbf{V}.\nabla)(\rho+p) + \gamma^4(2\mathbf{V}.\mathbf{v})V_i(\mathbf{V}.\nabla)(\rho+p) - \frac{1}{4\pi\alpha}\{(\alpha\delta B_i)_{,j}$$
$$-(\alpha\delta B_j)_{,i}\}B^j = -(\delta p)_i - \gamma^2\{(\delta\rho+\delta p) + 2(\rho+p)\gamma^2(\mathbf{V}.\mathbf{v})\}a_i$$
$$+\frac{1}{4\pi\alpha}\{(\alpha B_i)_{,j} - (\alpha B_j)_{,i}\}\delta B^j - (\rho+p)\gamma^4(v_i V^j + v^j V_i)V_{k,j}V^k$$
$$-\gamma^2\{(\delta\rho+\delta p)V^j + 2(\rho+p)\gamma^2(\mathbf{V}.\mathbf{v})V^j + (\rho+p)v^j\}V_{i,j}$$
$$-\gamma^4 V_i\{(\delta\rho+\delta p)V^j + 4(\rho+p)\gamma^2(\mathbf{V}.\mathbf{v})V^j + (\rho+p)v^j\}V_{j,k}V^k, \tag{3.15}$$

$$\gamma^2\frac{1}{\alpha}\frac{\partial(\delta\rho+\delta p)}{\partial t} + \mathbf{v}.\nabla(\rho+p)\gamma^2 - \frac{1}{\alpha}\frac{\partial(\delta p)}{\partial t} + (\mathbf{V}.\nabla)(\delta\rho+\delta p)\gamma^2$$
$$+2(\rho+p)\gamma^4(\mathbf{V}.\nabla)(\mathbf{V}.\mathbf{v}) + 2(\rho+p)\gamma^2(\mathbf{v}.\mathbf{a}) + 4(\rho+p)\gamma^4(\mathbf{V}.\mathbf{v})$$
$$(\mathbf{V}.\mathbf{a}) + 2(\delta\rho+\delta p)\gamma^2(\mathbf{V}.\mathbf{a}) + (\rho+p)\gamma^2(\nabla.\mathbf{v}) + 2(\rho+p)\gamma^4$$
$$(\mathbf{V}.\mathbf{v})(\nabla.\mathbf{V}) + (\delta\rho+\delta p)\gamma^2(\nabla.\mathbf{V}) = \frac{1}{4\pi\alpha}[\mathbf{v}.(\mathbf{B}.\frac{\partial\mathbf{B}}{\partial t})\mathbf{V} + \mathbf{V}.(\mathbf{B}.\frac{\partial\mathbf{B}}{\partial t})\mathbf{v}$$

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$$+\mathbf{V}.(\mathbf{B}.\delta\mathbf{B})\mathbf{V} + \mathbf{V}.(\delta\mathbf{B}\frac{\partial\mathbf{B}}{\partial t})\mathbf{V} - \mathbf{v}.(\mathbf{B}.\mathbf{V})\frac{\partial\mathbf{B}}{\partial t} - \mathbf{V}.(\mathbf{B}.\mathbf{V})\frac{\partial\delta\mathbf{B}}{\partial t}$$

$$-\mathbf{V}.(\mathbf{B}.\mathbf{v})\frac{\partial\delta\mathbf{B}}{\partial t} - \mathbf{V}.(\delta\mathbf{B}.\mathbf{V})\frac{\partial\mathbf{B}}{\partial t}] - \frac{1}{4\pi\alpha}[\mathbf{V}.(\mathbf{B}.\mathbf{B})\frac{\partial\mathbf{v}}{\partial t} - \mathbf{V}.(\mathbf{B}.\frac{\partial\delta\mathbf{v}}{\partial t})\mathbf{B}]$$

$$+\frac{1}{4\pi}[(\mathbf{v}\times\mathbf{B} + \mathbf{V}\times\delta\mathbf{B}).(\nabla\times\mathbf{B}) + (\mathbf{V}\times\mathbf{B}).(\nabla\times\delta\mathbf{B})] \qquad (3.16)$$

The component form of these equations, using Eq.(3.11), become

$$\frac{1}{\alpha}\frac{\partial b_x}{\partial t} - ub_{x,z} = (ub_x - Vb_z - v_x + \lambda v_z)\nabla\ln\alpha$$

$$-(v_{x,z} - \lambda v_{z,z} - \lambda'v_z + V'b_z + Vb_{z,z} - u'b_x), \qquad (3.17)$$

$$\frac{1}{\alpha}\frac{\partial b_z}{\partial t} = 0, \qquad (3.18)$$

$$b_{z,z} = 0, \qquad (3.19)$$

$$\rho\frac{1}{\alpha}\frac{\partial\tilde{\rho}}{\partial t} + p\frac{1}{\alpha}\frac{\partial\tilde{p}}{\partial t} + (\rho+p)\gamma^2 V(\frac{1}{\alpha}\frac{\partial v_x}{\partial t} + uv_{x,z}) + (\rho+p)\gamma^2 u$$

$$\times\frac{1}{\alpha}\frac{\partial v_z}{\partial t} + (\rho+p)(1+\gamma^2 u^2)v_{z,z} = -\gamma^2 u(\rho+p)[(1+2\gamma^2 V^2)V'$$

$$+2\gamma^2 uVu']v_x + (\rho+p)[(1-2\gamma^2 u^2)(1+\gamma^2 u^2)\frac{u'}{u}$$

$$-2\gamma^4 u^2 VV']v_z, \qquad (3.20)$$

$$\left\{(\rho+p)\gamma^2(1+\gamma^2 V^2) + \frac{B^2}{4\pi}\right\}\frac{1}{\alpha}\frac{\partial v_x}{\partial t} + \left\{(\rho+p)\gamma^4 uV - \frac{\lambda B^2}{4\pi}\right\}$$

$$\times\frac{1}{\alpha}\frac{\partial v_z}{\partial t} + \left\{(\rho+p)\gamma^2(1+\gamma^2 V^2) + \frac{B^2}{4\pi}\right\}uv_{x,z} + \left\{(\rho+p)\gamma^4 uV\right.$$

$$\left. -\frac{\lambda B^2}{4\pi}\right\}uv_{z,z} - \frac{B^2}{4\pi}(1+u^2)b_{x,z} - \frac{B^2}{4\pi\alpha}\left\{\alpha'(1+u^2) + \alpha uu'\right\}b_x$$

$$+\gamma^2 u(\rho\tilde{\rho} + p\tilde{p})\left\{(1+\gamma^2 V^2)V' + \gamma^2 uVu'\right\} + \gamma^2 uV(\rho'\tilde{\rho} + \rho\tilde{\rho}'$$

$$+p'\tilde{p} + p\tilde{p}') + [(\rho+p)\gamma^4 u\left\{(1+4\gamma^2 V^2)uu' + 4VV'(1+\gamma^2 V^2)\right\}$$

$$+\frac{B^2 u\alpha'}{4\pi\alpha} + \gamma^2 u(1+2\gamma^2 V^2)(\rho'+p')]v_x + [(\rho+p)\gamma^2\left\{(1+2\gamma^2 u^2)\right.$$

$$(1+2\gamma^2 V^2)V' - \gamma^2 V^2 V' + 2\gamma^2(1+2\gamma^2 u^2)uVu'\left.\right\} - \frac{B^2 u}{4\pi\alpha}(\lambda\alpha)'$$

$$+\gamma^2 V(1+2\gamma^2 u^2)(\rho'+p')]v_z = 0, \qquad (3.21)$$

$$\left\{(\rho+p)\gamma^2(1+\gamma^2u^2)+\frac{\lambda^2B^2}{4\pi}\right\}\frac{1}{\alpha}\frac{\partial v_z}{\partial t}+\left\{(\rho+p)\gamma^4uV-\frac{\lambda B^2}{4\pi}\right\}$$

$$\times\frac{1}{\alpha}\frac{\partial v_x}{\partial t}+\left\{(\rho+p)\gamma^2(1+\gamma^2u^2)+\frac{\lambda^2B^2}{4\pi}\right\}uv_{z,z}+\left\{(\rho+p)\gamma^4uV\right.$$

$$\left.-\frac{\lambda B^2}{4\pi}\right\}uv_{x,z}+\frac{\lambda B^2}{4\pi}(1+u^2)b_{x,z}+\frac{B^2}{4\pi\alpha}\left\{(\alpha\lambda)'-\alpha'\lambda+u\lambda(u\alpha'\right.$$

$$\left.+u'\alpha)\right\}b_x+(\rho\tilde\rho+p\tilde p)\gamma^2\left\{a_z+uu'(1+\gamma^2u^2)+\gamma^2u^2VV'\right\}$$

$$+(1+\gamma^2u^2)(p'\tilde p+p\tilde p')+\gamma^2u^2(\rho'\tilde\rho+\rho\tilde\rho')+[(\rho+p)\gamma^4$$

$$\times\left\{u^2V'(1+4\gamma^2V^2)+2V(a_z+uu'(1+2\gamma^2u^2))\right\}-\frac{\lambda B^2u\alpha'}{4\pi\alpha}$$

$$+2\gamma^4u^2V(\rho'+p')]v_x+[(\rho+p)\gamma^2\left\{u'(1+\gamma^2u^2)(1+4\gamma^2u^2)\right.$$

$$\left.+2u\gamma^2(a_z+(1+2\gamma^2u^2)VV')\right\}+\frac{\lambda B^2u}{4\pi\alpha}(\alpha\lambda)'+2\gamma^2u(1$$

$$+\gamma^2u^2)(\rho'+p')]v_z=0,\qquad\qquad(3.22)$$

$$\frac{1}{\alpha}\gamma^2\rho\frac{\partial\tilde\rho}{\partial t}+\frac{1}{\alpha}\gamma^2p\frac{\partial\tilde p}{\partial t}+\gamma^2(\rho'+p')v_z+u\gamma^2(\rho\tilde\rho_{,z}+p\tilde p_{,z}+\rho'\tilde\rho$$

$$+p'\tilde p)-\frac{1}{\alpha}p\frac{\partial\tilde p}{\partial t}+2\gamma^2u(\rho\tilde\rho+p\tilde p)a_z+\gamma^2u'(\rho\tilde\rho+p\tilde p)+2(\rho$$

$$+p)\gamma^4(uV'+2uVa_z+u'V)v_x+2(\rho+p)\gamma^2(2\gamma^2uu'+a_z\gamma^4$$

$$+2\gamma^2u^2a_z)v_z+2(\rho+p)\gamma^4uVv_{x,z}+(\rho+p)\gamma^2(1+2\gamma^2u^2)$$

$$\times v_{z,z}-\frac{B^2}{4\pi\alpha}[(V^2+u^2)\lambda b_x+(V^2+u^2)b_z-\lambda V(\lambda V$$

$$+u)\frac{\partial b_x}{\partial t}-u(\lambda V+u)\frac{\partial b_z}{\partial t}]-\frac{B^2}{4\pi\alpha}[(V-\lambda u)v_{x,t}+\lambda(u\lambda$$

$$-V)v_{z,t}]+\frac{B^2}{4\pi}(\lambda\lambda'v_z-\lambda'v_x-\lambda'Vb_z$$

$$+\lambda'ub_x-Vb_{x,z}+u\lambda b_{x,z})=0.\qquad\qquad(3.23)$$

For the purpose of Fourier analysis, the following harmonic spacetime dependence of perturbation is assumed

$$\widetilde\rho(t,z)=c_1e^{-\iota(\omega t-kz)},\qquad\widetilde p(t,z)=c_2e^{-\iota(\omega t-kz)},$$
$$v_z(t,z)=c_3e^{-\iota(\omega t-kz)},\qquad v_x(t,z)=c_4e^{-\iota(\omega t-kz)},$$
$$b_z(t,z)=c_5e^{-\iota(\omega t-kz)},\qquad b_x(t,z)=c_6e^{-\iota(\omega t-kz)}.\qquad(3.24)$$

Here k and ω are the z-component of the wave vector $(0,0,k)$ and angular frequency respectively. Plasma wave properties near the event horizon can

be explored by the wave vector which is also used to obtain refractive index. We define wave vector and refractive index as follows:

- **Wave Vector**: The direction in which a plane wave propagates is represented by a wave vector. Its magnitude gives the wave number.

- **Refractive Index**: When light travels from one medium to another (usually from vacuum) then its ratio between the two mediums is given by the refractive index. The change in the refractive index with respect to angular frequency decides whether the dispersion will be normal or anomalous.

Using Eq.(3.24) in Eqs.(3.17)-(3.23), we get their Fourier analyzed form

$$c_4(\alpha' + \iota k\alpha) - c_3 \ \{(\alpha\lambda)' + \iota k\alpha\lambda \ \} - c_5(\alpha V)' - c_6\{(\alpha u)' + \iota\omega$$
$$+\iota k u\alpha\} = 0, \tag{3.25}$$

$$c_5(\frac{-\iota\omega}{\alpha}) = 0, \tag{3.26}$$

$$c_5 \iota k = 0, \tag{3.27}$$

$$c_1(\frac{-\iota\omega}{\alpha}\rho) + c_2(\frac{-\iota\omega}{\alpha}p) + c_3(\rho + p)[\frac{-\iota\omega}{\alpha}\gamma^2 u + (1 + \gamma^2 u^2)\iota k$$
$$-(1 - 2\gamma^2 u^2)(1 + \gamma^2 u^2)\frac{u'}{u} + 2\gamma^4 u^2 V V'] + c_4(\rho + p)\gamma^2[(\frac{-\iota\omega}{\alpha}$$
$$+\iota k u)V + u(1 + 2\gamma^2 V^2)V' + 2\gamma^2 u^2 V u'] = 0, \tag{3.28}$$

$$c_1[\rho\gamma^2 u\{(1 + \gamma^2 V^2)V' + \gamma^2 V u u'\} + \gamma^2 V u(\rho' + \iota k\rho)]$$
$$+c_2[p\gamma^2 u\{(1 + \gamma^2 V^2)V' + \gamma^2 V u u'\} + \gamma^2 V u(p' + \iota k p)]$$
$$+c_3[(\rho + p)\gamma^2\{(1 + 2\gamma^2 u^2)(1 + 2\gamma^2 V^2)V' + (\frac{-\iota\omega}{\alpha} + \iota k u)\gamma^2 V u$$
$$-\gamma^2 V^2 V' + 2\gamma^2(1 + 2\gamma^2 u^2)u V u'\} + \gamma^2 V(1 + 2\gamma^2 u^2)(\rho' + p')$$
$$-\frac{B^2 u}{4\pi\alpha}(\lambda\alpha)' + \frac{\lambda B^2}{4\pi}(\frac{\iota\omega}{\alpha} - \iota k u)] + c_4[(\rho + p)\gamma^4 u\{(1 + 4\gamma^2 V^2)$$
$$\times u u' + 4 V V'(1 + \gamma^2 V^2)\} + (\rho + p)\gamma^2(1 + \gamma^2 V^2)(\frac{-\iota\omega}{\alpha} + \iota k u)$$
$$+\gamma^2 u(1 + 2\gamma^2 V^2)(\rho' + p') + \frac{B^2 u\alpha'}{4\pi\alpha} - \frac{B^2}{4\pi}(\frac{\iota\omega}{\alpha} - \iota k u)]$$
$$-c_6\frac{B^2}{4\pi\alpha}[\alpha u u' + \alpha'(1 + u^2) + (1 + u^2)\iota k\alpha] = 0, \tag{3.29}$$

$$c_1[\rho\gamma^2\{a_z + (1+\gamma^2 u^2)uu' + \gamma^2 u^2 VV'\} + \gamma^2 u^2(\rho' + \iota k\rho)]$$
$$+c_2[p\gamma^2\{a_z + (1+\gamma^2 u^2)uu' + \gamma^2 u^2 VV'\} + (1+\gamma^2 u^2)$$
$$\times(p' + \iota kp)] + c_3[(\rho+p)\gamma^2\{(1+\gamma^2 u^2)(\frac{-\iota\omega}{\alpha} + \iota ku)$$
$$+u'(1+\gamma^2 u^2)(1+4\gamma^2 u^2) + 2u\gamma^2(a_z + (1+2\gamma^2 u^2)VV')\}$$
$$+2\gamma^2 u(1+\gamma^2 u^2)(\rho'+p') + \frac{\lambda B^2 u}{4\pi\alpha}(\lambda\alpha)' - \frac{\lambda^2 B^2}{4\pi}(\frac{\iota\omega}{\alpha} - \iota ku)]$$
$$+c_4[(\rho+p)\gamma^4\{(\frac{-\iota\omega}{\alpha} + \iota ku)uV + u^2 V'(1+4\gamma^2 V^2) + 2V(a_z$$
$$+(1+2\gamma^2 u^2)uu')\} + 2\gamma^4 u^2 V(\rho'+p') + \frac{\lambda B^2}{4\pi}(\frac{\iota\omega}{\alpha} - \iota ku)$$
$$-\frac{\lambda B^2 u\alpha'}{4\pi\alpha}] + c_6[\frac{B^2}{4\pi\alpha}\{-(\lambda\alpha)' + \alpha'\lambda - u\lambda(u\alpha' + u'\alpha)\}$$
$$+\frac{\lambda B^2}{4\pi}(1+u^2)\iota k] = 0, \tag{3.30}$$
$$c_1\{(\frac{-\iota\omega}{\alpha}\gamma^2 + \iota ku\gamma^2 + 2u\gamma^2 a_z + \gamma^2 u')\rho + u\rho'\gamma^2\} + c_2\{(\frac{\iota\omega}{\alpha}(1-\gamma^2)$$
$$+\iota ku\gamma^2 + 2\gamma^2 ua_z + \gamma^2 u')p + u\gamma^2 p'\} + c_3\gamma^2\{(\rho'+p') + 2$$
$$\times(2\gamma^4 uu' + a_z + 2\gamma^2 u^2 a_z)(\rho+p) + (1+2\gamma^2 u^2)(\rho+p)\iota k + \frac{\lambda B^2}{4\pi\alpha}$$
$$\times(\lambda u - V)\iota\omega + \alpha\lambda'\} + c_4[2(\rho+p)\gamma^2\{(uV' + 2uVa_z + u'V) + uV\iota k\}$$
$$+\frac{B^2}{4\pi\alpha}(V - u\lambda)\iota\omega - \alpha\lambda'] + c_6[\frac{-B^2}{4\pi\alpha}\{(V^2 + u^2)\lambda + \lambda V(\lambda V + u)\iota\omega\}$$
$$-\alpha\lambda' u + \iota k\alpha(V - u\lambda)] = 0. \tag{3.31}$$

Dispersion relations will be obtained by using these equations.

4 Rotating Non-Magnetized Flow with Hot Plasma

In this section, rotating non-magnetized background of plasma flow is assumed, i.e., $\mathbf{B} = 0$. Thus, the evolution equations (3.2) and (3.3) of magnetic field are satisfied. Substituting $B = 0 = \lambda$ and $c_5 = 0 = c_6$ in the Fourier

analyzed perturbed GRMHD equations (Eqs.(3.28)-(3.31)), we have

$$c_1(\frac{-\iota\omega}{\alpha}\rho) + c_2(\frac{-\iota\omega}{\alpha}p) + c_3(\rho+p)[\frac{-\iota\omega}{\alpha}\gamma^2 u + (1+\gamma^2 u^2)\iota k$$

$$-(1-2\gamma^2 u^2)(1+\gamma^2 u^2)\frac{u'}{u} + 2\gamma^4 u^2 V V'] + c_4(\rho+p)\gamma^2[(\frac{-\iota\omega}{\alpha}$$

$$+\iota k u)V + u(1+2\gamma^2 V^2)V' + 2\gamma^2 u^2 V u'] = 0, \qquad (4.1)$$

$$c_1[\rho\gamma^2 u\{(1+\gamma^2 V^2)V' + \gamma^2 V u u'\} + \gamma^2 V u(\rho' + \iota k \rho)] + c_2[p\gamma^2 u$$

$$\times\{(1+\gamma^2 V^2)V' + \gamma^2 V u u'\} + \gamma^2 V u(p' + \iota k p)] + c_3[(\rho+p)\gamma^2$$

$$\times\{(\frac{-\iota\omega}{\alpha} + \iota k u)\gamma^2 V u + (1+2\gamma^2 u^2)(1+2\gamma^2 V^2)V' - \gamma^2 V^2 V'$$

$$+2\gamma^2(1+2\gamma^2 u^2)u V u'\} + \gamma^2 V(1+2\gamma^2 u^2)(\rho'+p')] + c_4[(\rho+p)$$

$$\{\gamma^2(1+\gamma^2 V^2)(\frac{-\iota\omega}{\alpha} + \iota k u) + \gamma^4 u((1+4\gamma^2 V^2)u u' + 4 V V'(1+$$

$$\gamma^2 V^2))\} + \gamma^2 u(1+2\gamma^2 V^2)(\rho'+p')] = 0, \qquad (4.2)$$

$$c_1[\rho\gamma^2\{a_z + (1+\gamma^2 u^2)u u' + \gamma^2 u^2 V V'\} + \gamma^2 u^2(\rho' + \iota k \rho)]$$

$$+c_2[p\gamma^2\{a_z + (1+\gamma^2 u^2)u u' + \gamma^2 u^2 V V'\} + (p' + \iota k p)\times$$

$$(1+\gamma^2 u^2)] + c_3[(\rho+p)\gamma^2\{(1+\gamma^2 u^2)(\frac{-\iota\omega}{\alpha} + \iota k u) + u'(1+\gamma^2 u^2)$$

$$\times(1+4\gamma^2 u^2) + 2u\gamma^2(a_z + (1+2\gamma^2 u^2)V V')\} + 2\gamma^2 u(1+\gamma^2 u^2)$$

$$\times(\rho'+p')] + c_4[(\rho+p)\gamma^4\{(\frac{-\iota\omega}{\alpha} + \iota k u)u V + u^2 V'(1+4\gamma^2 V^2)$$

$$+2V(a_z + (1+2\gamma^2 u^2)u u')\} + 2\gamma^4 u^2 V(\rho'+p')] = 0, \qquad (4.3)$$

$$c_1\{(\frac{-\iota\omega}{\alpha}\gamma^2 + \iota k u \gamma^2 + 2u\gamma^2 a_z + \gamma^2 u')\rho + u\rho'\gamma^2\} + c_2\{(\frac{\iota\omega}{\alpha}(1-\gamma^2)$$

$$+\iota k u \gamma^2 + 2\gamma^2 u a_z + \gamma^2 u')p + u\gamma^2 p'\} + c_3\gamma^2\{(\rho'+p') + 2(2\gamma^2 u u'$$

$$+a_z + 2\gamma^2 u^2 a_z)(\rho+p) + (1+2\gamma^2 u^2)(\rho+p)\iota k\} + c_4(\rho+p)2\gamma^4$$

$$\{(u V' + 2u V a_z + u' V) + u V \iota k\} = 0. \qquad (4.4)$$

4.1 Numerical Solutions

For the rotating non-magnetized plasma, we use the following assumptions to find out the numerical solutions

- Specific enthalpy: $\mu = \sqrt{\frac{1-(\tanh(10z)/10)^2}{2}}$,

- Time lapse: $\alpha = \tanh(10z)/10$,

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- Stationary fluid: $\alpha\gamma = 1$ with velocity components $V = u$ gives the following relation: $\alpha\gamma = 1 \Rightarrow \gamma = 1/\sqrt{1 - u^2 - V^2} = 1/\alpha$.

- Velocity components: $u = V$, x and z-components of velocity yield $u = V = -\sqrt{\frac{1-\alpha^2}{2}}$.

- Stiff fluid: $\rho = p = \mu/2$.

These assumptions satisfy the GRMHD equations (Eqs.(3.2)-(3.6)) for the region $1.4 \leq z \leq 10, 0 \leq \omega \leq 10$. A complex dispersion relation [34] is obtained by solving the determinant of the coefficients of constants of Eqs.(4.1)-(4.4). We obtain a quartic equation in k from the real part of the determinant

$$A_1(z)k^4 + A_2(z,\omega)k^3 + A_3(z,\omega)k^2 + A_4(z,\omega)k + A_5(z,\omega) = 0 \qquad (4.5)$$

which yields four values of k out of which two are real and two complex conjugate. The imaginary part gives a cubic equation in k

$$B_1(z)k^3 + B_2(z,\omega)k^2 + B_3(z,\omega)k + B_4(z,\omega) = 0 \qquad (4.6)$$

from which one real value of k is obtained and the remaining two are complex conjugate of each other. From the real values of k ((4.5) and (4.6)), wave vector, refractive index and its change with respect to angular frequency are shown in Figures **1-3**. The results deduced from these figures can be displayed in the following table.

Table I. Direction and refractive index of waves

	Direction of Waves	**Refractive Index** (n)
1	Move towards the event horizon	$n < 1$ and decreases in the region $1.4 \leq z \leq 1.8, 0 \leq \omega \leq 10$ with the decrease in z
2	Move away from the event horizon	$n < 1$ and decreases in the region $1.4 \leq z \leq 1.6, 0 \leq \omega \leq 10$ with the decrease in z
3	Move away from the event horizon	$n < 1$ and decreases in the region $1.4 \leq z \leq 1.7, 0 \leq \omega \leq 10$ with the decrease in z

Figures **1** and **2** indicate normal and anomalous dispersion of waves at random points while Figure **3** gives normal dispersion in the whole region $1.4 \leq z \leq 10, 0 \leq \omega \leq 10$. The group and phase velocities are equal in magnitudes but opposite in directions in all figures.
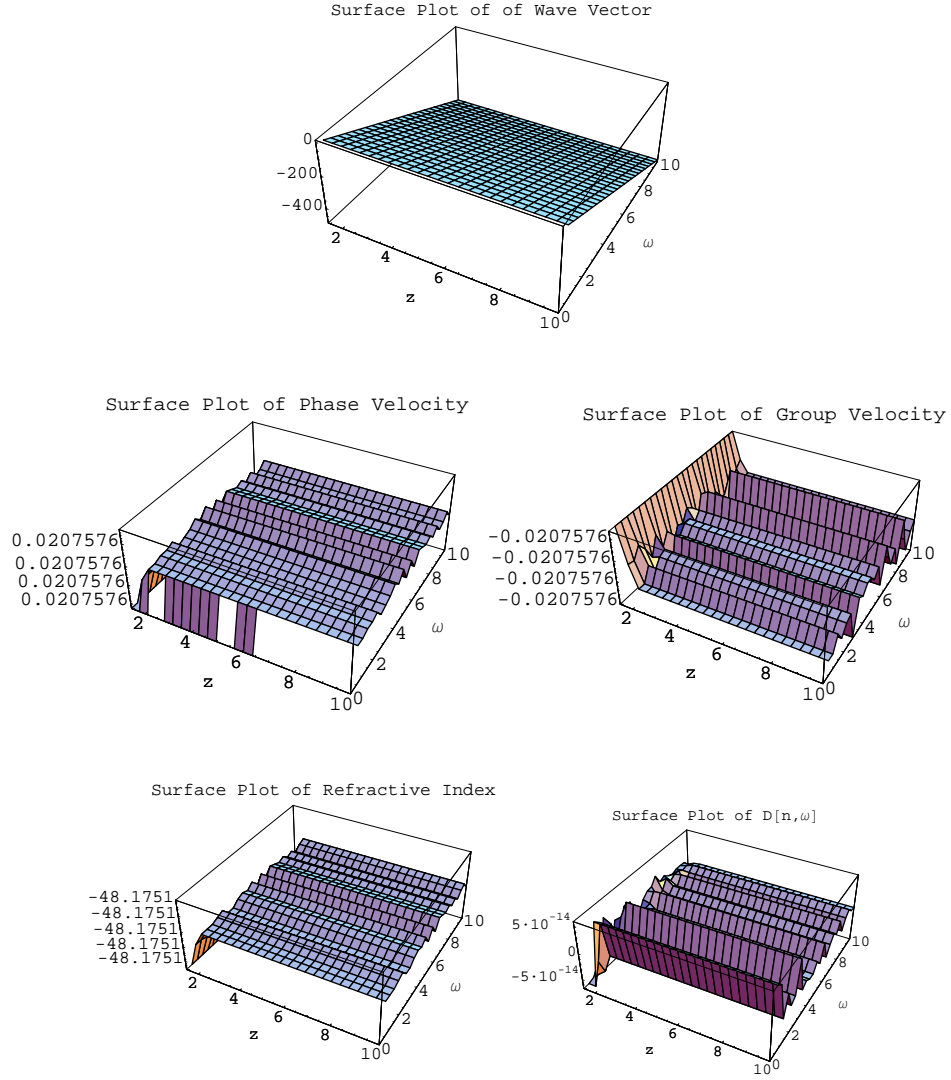
Surface Plot of of Wave Vector

Surface Plot of Phase Velocity

Surface Plot of Group Velocity

Surface Plot of Refractive Index

Surface Plot of D[n,ω]

Figure 1: Waves are directed towards the event horizon. The dispersion is found to be normal as well as anomalous randomly.

Surface Plot of of Wave Vector

Surface Plot of Phase Velocity

Surface Plot of Group Velocity

Surface Plot of Refractive Index

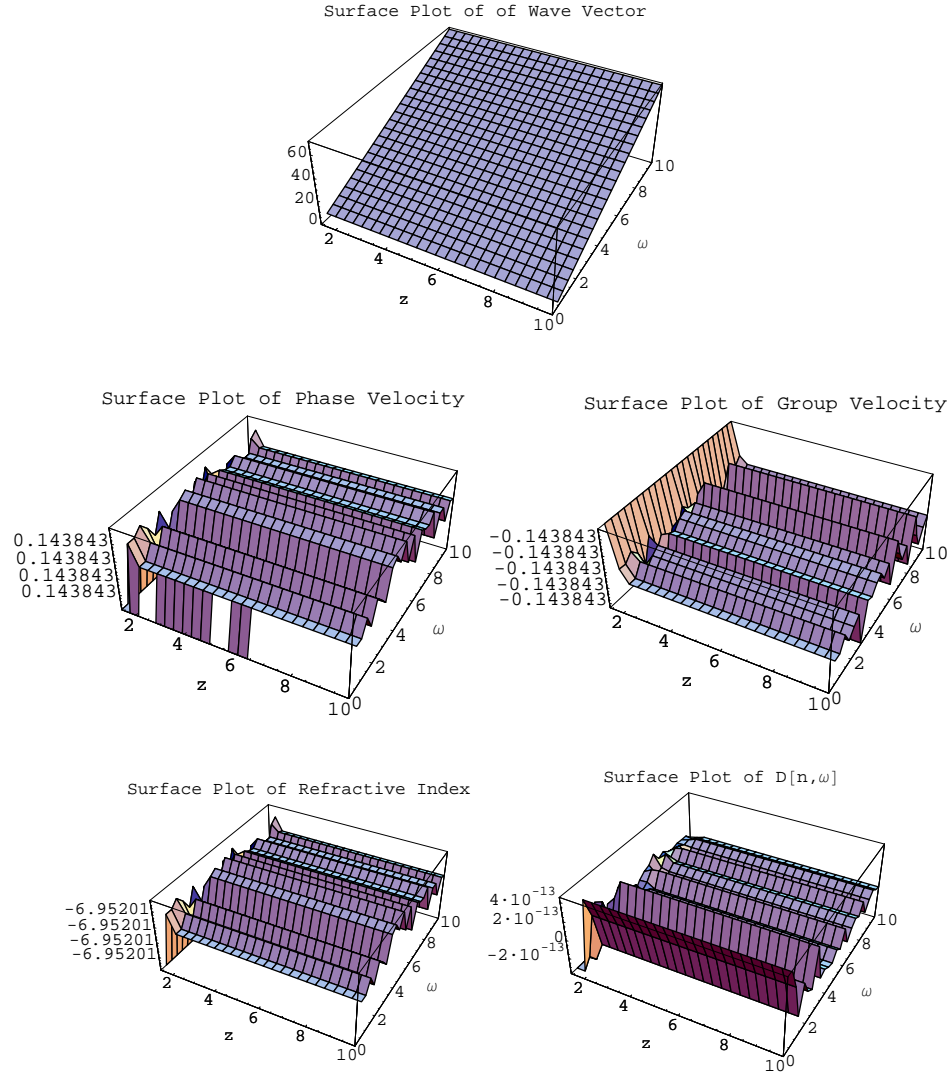Surface Plot of D[n,ω]

Figure 2: Waves are directed away from the event horizon. Region has normal and anomalous dispersion at random points.
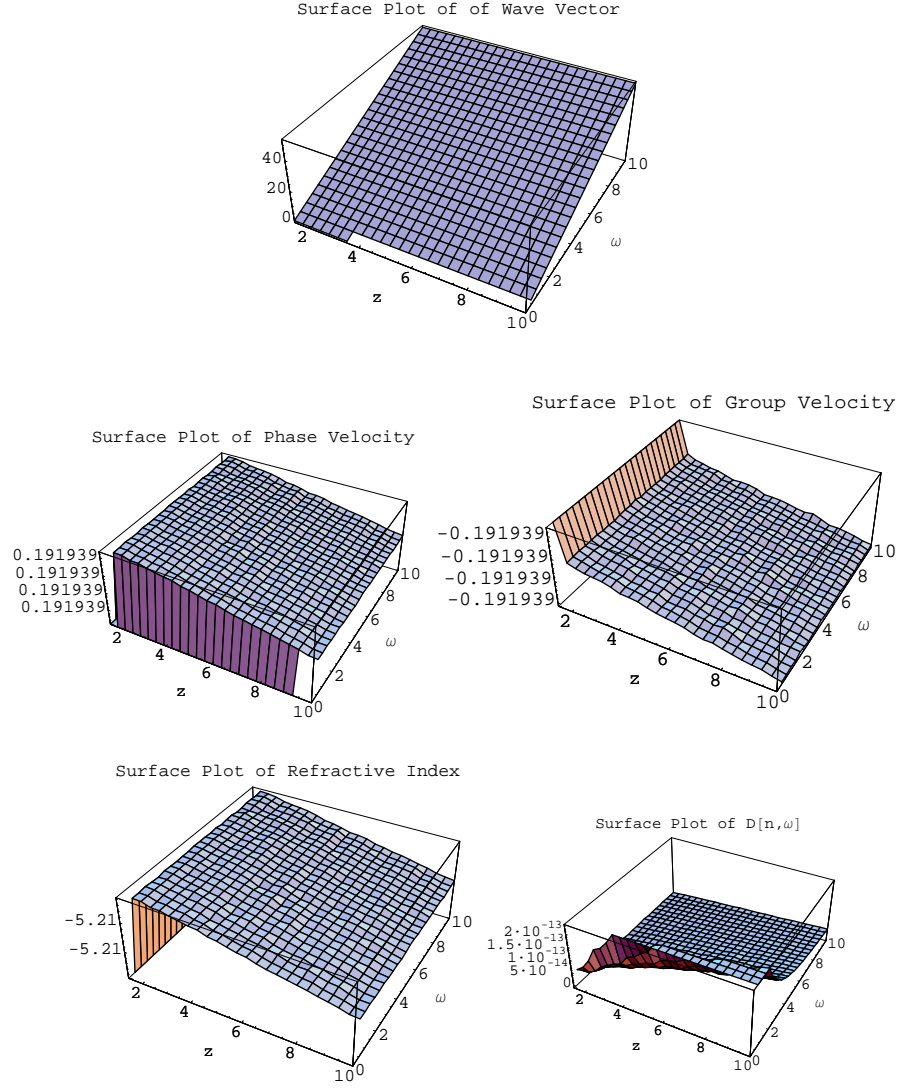
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Figure 3: Waves move away from the event horizon. The whole region has normal dispersion of waves.

5   Rotating Magnetized Flow with Hot Plasma

In this general case, plasma is assumed to be rotating and magnetized. Fluid's velocity and magnetic field are supposed to lie in xz-plane. The respective Fourier analyzed perturbed GRMHD equations, i.e., Eqs.(3.25)-(3.31) are given in Section 3.

5.1   Numerical Solutions

We consider the same assumptions for the values of lapse function, velocity, pressure, density and specific enthalpy as given in Section 4. The restrictions on the magnetic field are as follows

- $B = \sqrt{\frac{176}{7}}$.

- For $u = V$ and $V^F = 0$, Eq.(3.8) yields $\lambda = 1$.

The above restrictions satisfy the perfect GRMHD equations (Eqs.(3.2)-(3.6)) for the range $1.4 \leq z \leq 10$, $0 \leq \omega \leq 10$. We have $c_5 = 0$ from Eqs.(3.26)-(3.27). Using above assumptions in Eqs.(3.25) and (3.28)-(3.31), we obtain two dispersion relations. The real part gives

$$A_1(z)k^4 + A_2(z,\omega)k^3 + A_3(z,\omega)k^2 + A_4(z,\omega)k + A_5(z,\omega) = 0 \qquad (5.1)$$

yielding four values of k but all are imaginary. The dispersion relation obtained from imaginary part is

$$\begin{aligned} &B_1(z)k^5 + B_2(z,\omega)k^4 + B_3(z,\omega)k^3 + B_4(z,\omega)k^2 + B_5(z,\omega)k \\ &+B_6(z,\omega) = 0 \end{aligned} \qquad (5.2)$$

which gives two real values of k. Figures **4**-**8** represent their solutions. The following table shows the results obtained from these figures.

Table II. Direction and refractive index of waves

	Direction of Waves	Refractive Index (n)
4	Move towards the event horizon	$n < 1$ and increases in the region $1.5 \leq z \leq 1.7, 0 \leq \omega \leq 10$ with the decrease in z
5	Move away from the event horizon	$n < 1$ and decreases in the region $1.4 \leq z \leq 1.8, 0 \leq \omega \leq 10$ with the decrease in z
6	Move towards the event horizon	$n < 1$ and increases in the region $1.4 \leq z \leq 1.7, 0 \leq \omega \leq 10$ with the decrease in z
7	Move away from the event horizon	$n < 1$ and increases in the region $1.4 \leq z \leq 1.75, 0 \leq \omega \leq 10$ with the decrease in z
8	Move away from the event horizon	$n < 1$ and increases in the region $1.4 \leq z \leq 1.8, 0 \leq \omega \leq 10$ with the decrease in z

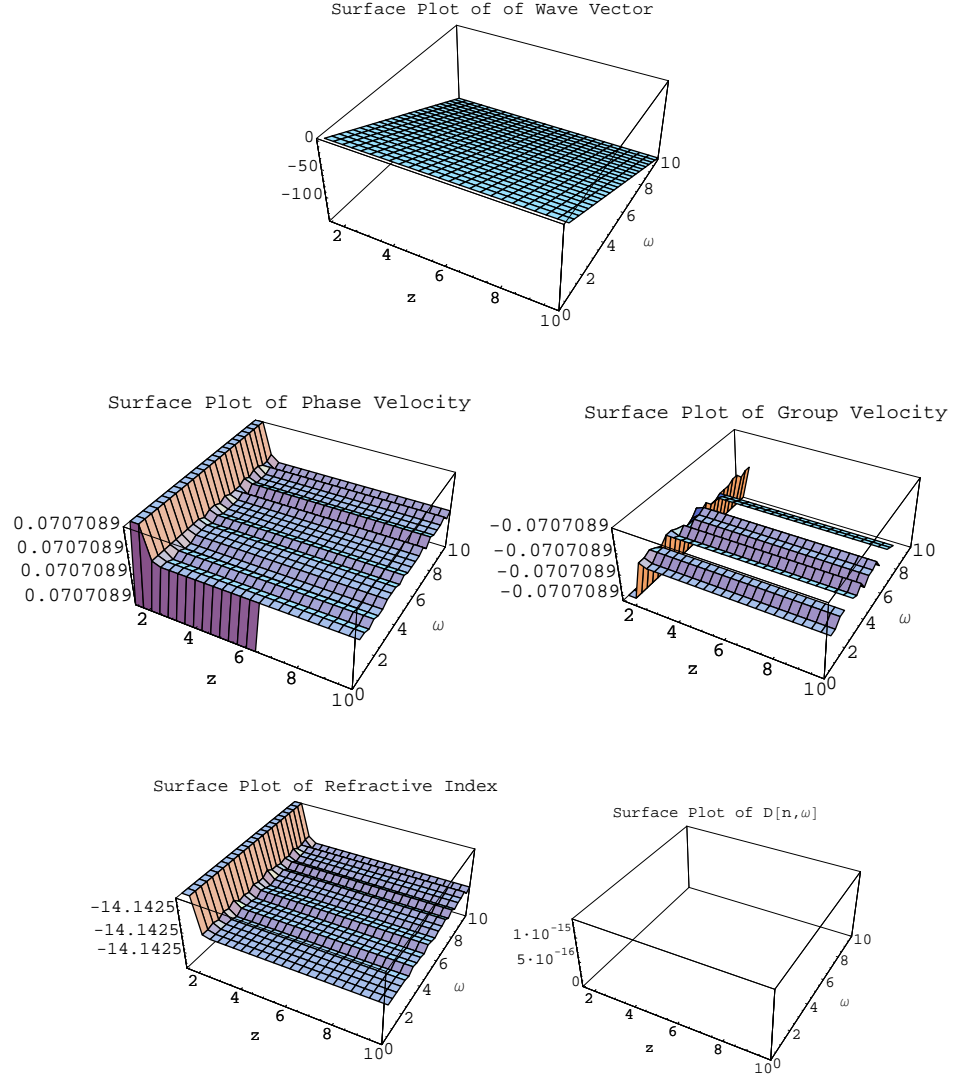These figures show that group and phase velocities are antiparallel.

Figure 4: Waves move towards the event horizon. The dispersion is found to be normal.

Surface Plot of Wave Vector

Surface Plot of Phase Velocity

Surface Plot of Group Velocity

Surface Plot of Refractive Index
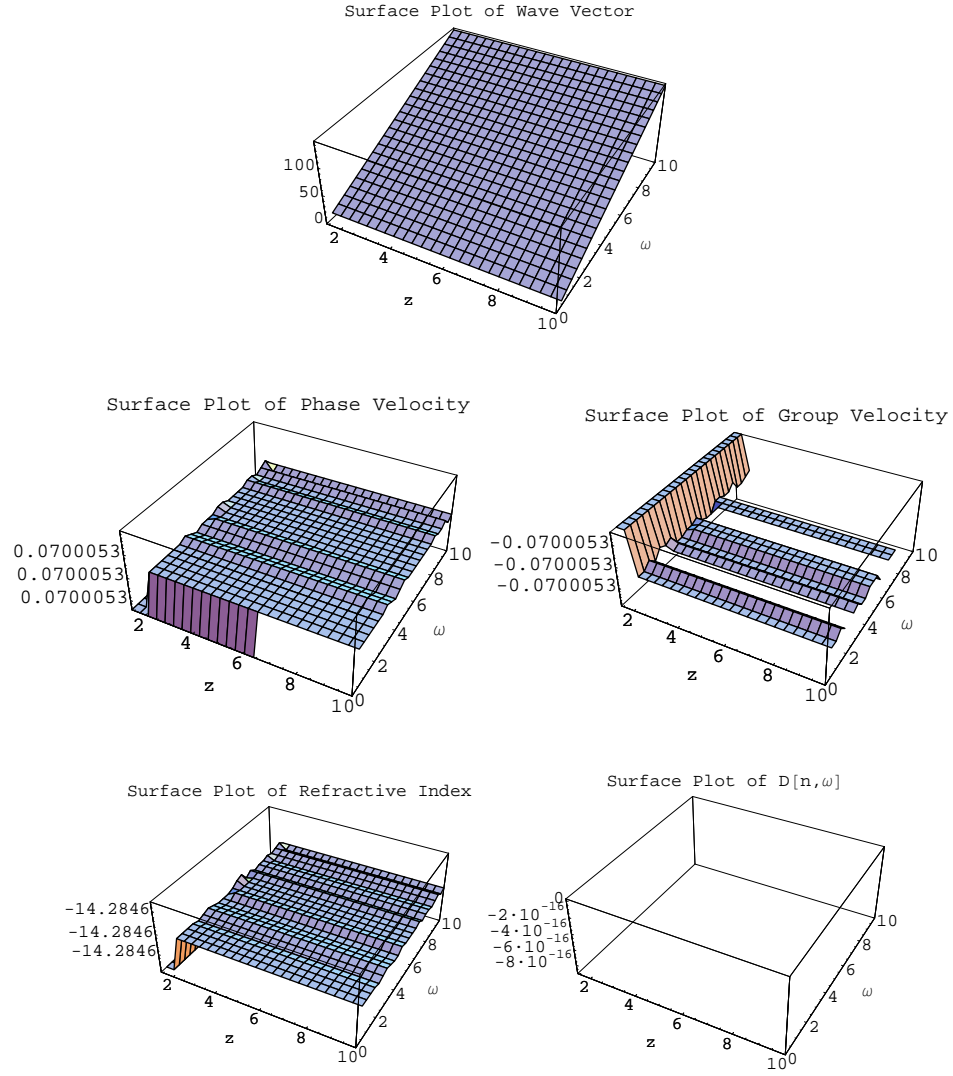
Surface Plot of D[n,ω]

Figure 5: Waves move away from the event horizon. Region has anomalous dispersion of waves.
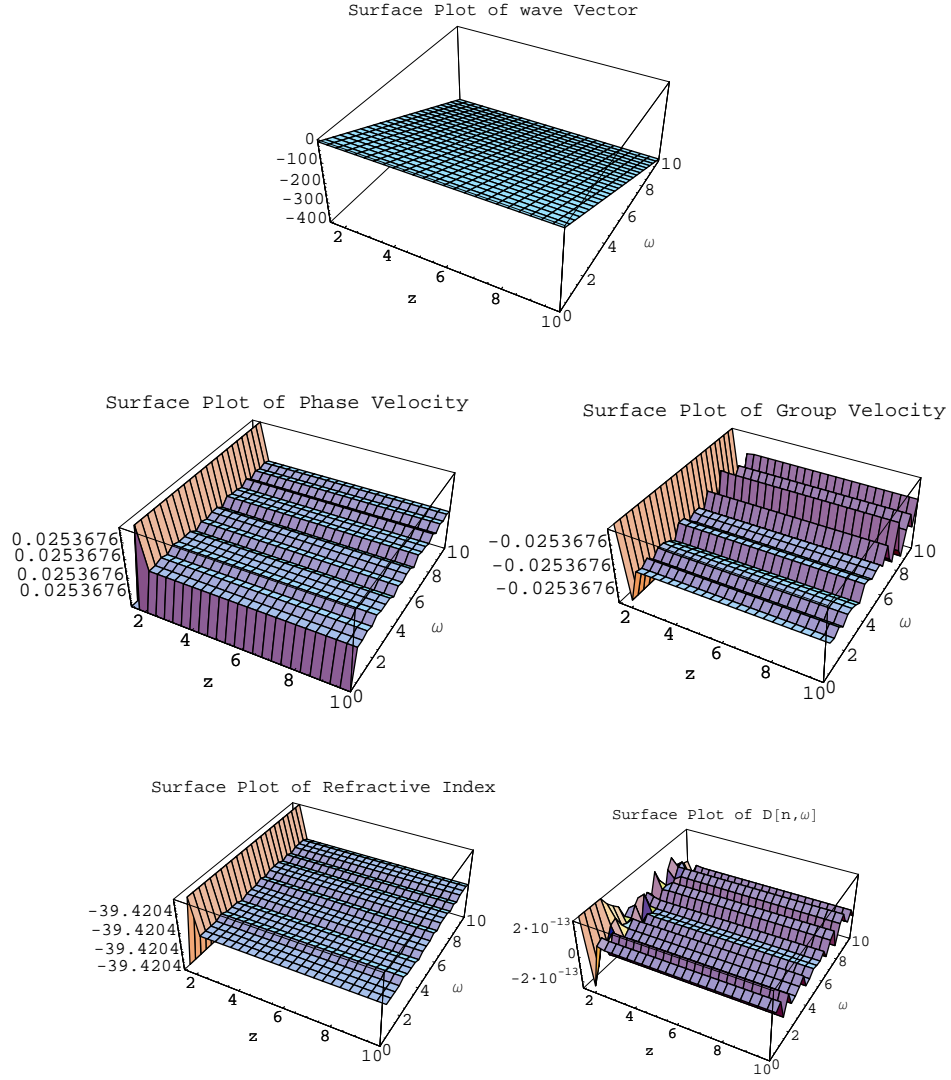
Figure 6: Waves move towards the event horizon. The dispersion is normal as well as anomalous at random points.
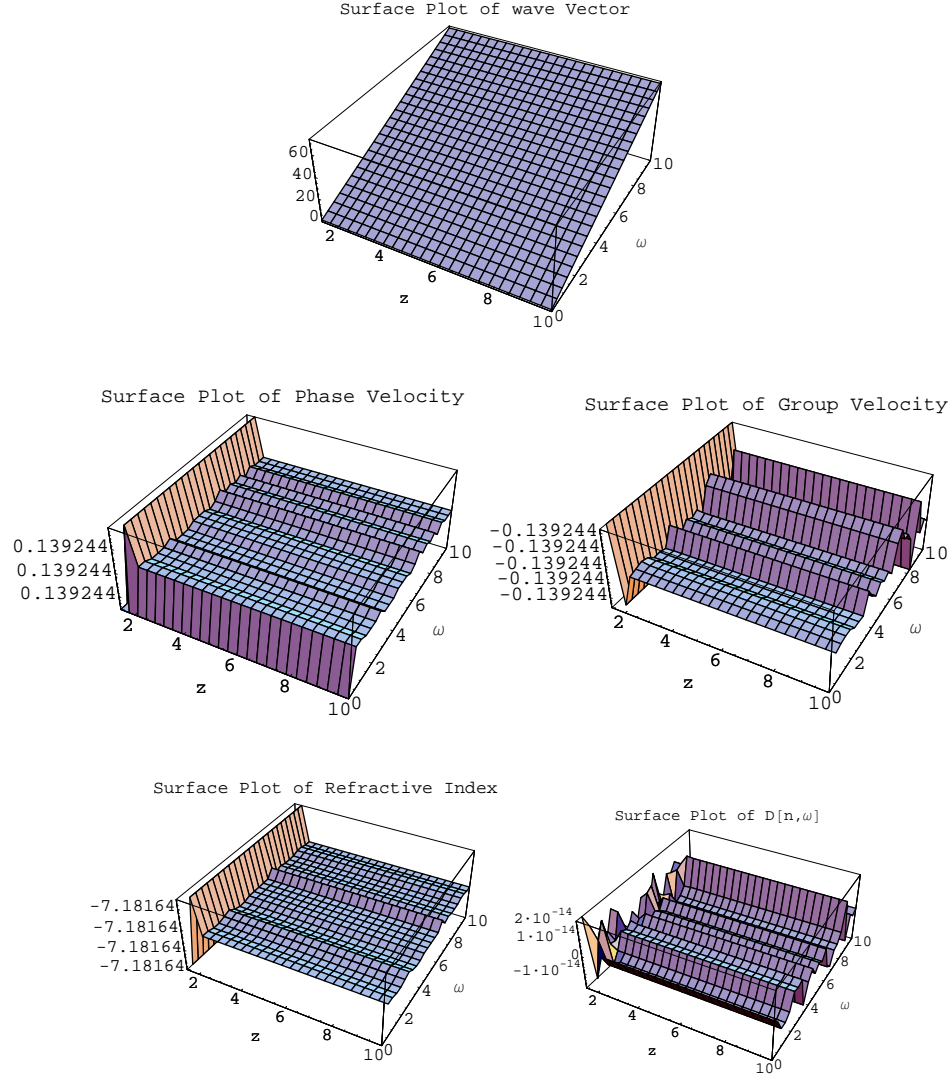
Figure 7: Waves move away from the event horizon. Region has normal and anomalous dispersion of waves randomly.
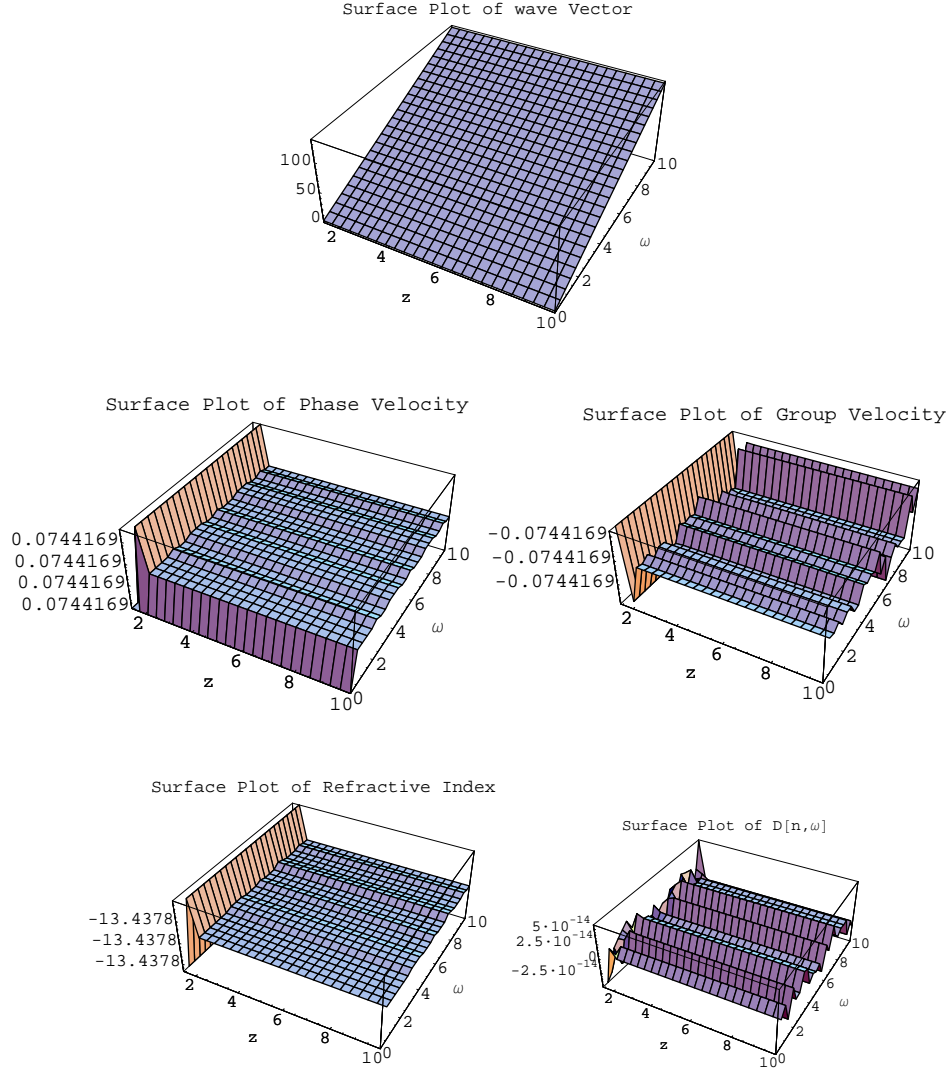
Figure 8: Waves move away from the event horizon. The dispersion is normal as well as anomalous at random points.

Normal and anomalous dispersion of waves can be classified into different regions given in the following table.

Table III. Regions of dispersion

	Normal dispersion	Anomalous dispersion
6	$2 \leq z \leq 10, 2.9 \leq \omega \leq 3$ $2 \leq z \leq 10, 3.2 \leq \omega \leq 3.6$ $2 \leq z \leq 10, 4.3 \leq \omega \leq 4.4$ $2 \leq z \leq 10, 5.1 \leq \omega \leq 5.3$ $2 \leq z \leq 10, 6.5 \leq \omega \leq 6.7$ $2 \leq z \leq 10, 7.1 \leq \omega \leq 7.3$ $2 \leq z \leq 10, 8.4 \leq \omega \leq 8.6$ $2 \leq z \leq 10, 9.4 \leq \omega \leq 9.6$	$2 \leq z \leq 10, 2.5 \leq \omega \leq 2.8$ $2 \leq z \leq 10, 3.7 \leq \omega \leq 3.8$ $2 \leq z \leq 10, 4.1 \leq \omega \leq 4.2$ $2 \leq z \leq 10, 5.4 \leq \omega \leq 5.6$ $2.1 \leq z \leq 10, 6.2 \leq \omega \leq 6.4$ $2 \leq z \leq 10, 7.4 \leq \omega \leq 7.6$ $2 \leq z \leq 10, 8.1 \leq \omega \leq 8.2$ $2 \leq z \leq 10, 9 \leq \omega \leq 9.2$
7	$2 \leq z \leq 10, 2 \leq \omega \leq 2.2$ $2 \leq z \leq 10, 3.5 \leq \omega \leq 3.6$ $2 \leq z \leq 10, 4.1 \leq \omega \leq 4.3$ $1.6 \leq z \leq 10, 5.1 \leq \omega \leq 5.2$ $2 \leq z \leq 10, 6 \leq \omega \leq 6.2$ $2 \leq z \leq 10, 7.7 \leq \omega \leq 7.8$ $2 \leq z \leq 10, 8.3 \leq \omega \leq 8.6$ $2 \leq z \leq 10, 9.2 \leq \omega \leq 9.4$	$2 \leq z \leq 10, 2.6 \leq \omega \leq 2.7$ $2 \leq z \leq 10, 3.2 \leq \omega \leq 3.4$ $1.6 \leq z \leq 10, 4.4 \leq \omega \leq 4.5$ $1.6 \leq z \leq 10, 5.3 \leq \omega \leq 5.4$ $1.6 \leq z \leq 10, 6.6 \leq \omega \leq 6.8$ $2 \leq z \leq 10, 7.1 \leq \omega \leq 7.2$ $2 \leq z \leq 10, 8 \leq \omega \leq 8.2$ $2 \leq z \leq 10, 9.5 \leq \omega \leq 9.9$
8	$1.5 \leq z \leq 10, 2 \leq \omega \leq 2.2$ $1.5 \leq z \leq 10, 3.5 \leq \omega \leq 3.6$ $1.6 \leq z \leq 10, 4 \leq \omega \leq 4.2$ $1.8 \leq z \leq 10, 5.6 \leq \omega \leq 5.7$ $2 \leq z \leq 10, 6 \leq \omega \leq 6.2$ $2 \leq z \leq 10, 7.7 \leq \omega \leq 7.8$ $2 \leq z \leq 10, 8.3 \leq \omega \leq 8.6$ $2 \leq z \leq 10, 9.2 \leq \omega \leq 9.4$	$1.5 \leq z \leq 10, 2.6 \leq \omega \leq 2.8$ $1.5 \leq z \leq 10, 3.2 \leq \omega \leq 3.3$ $1.6 \leq z \leq 10, 4.4 \leq \omega \leq 4.5$ $1.8 \leq z \leq 10, 5.2 \leq \omega \leq 5.4$ $2.1 \leq z \leq 10, 6.3 \leq \omega \leq 6.4$ $2 \leq z \leq 10, 7.1 \leq \omega \leq 7.2$ $2 \leq z \leq 10, 8 \leq \omega \leq 8.2$ $2 \leq z \leq 10, 9.5 \leq \omega \leq 9.9$

6   Summary

In this paper, wave properties of hot plasma in the vicinity of the Schwarzschild event horizon for a Veselago medium are analyzed. The $3+1$ GRMHD equations are reformulated for this unusual medium. Linear perturbation in $3+1$ perfect GRMHD equations is considered and their component form is derived. Dispersion relations are found by using Fourier analysis technique for the rotating non-magnetized and rotating magnetized plasmas.

In the rotating non-magnetized background, Figure **1** shows that waves move towards the event horizon while Figures **2** and **3** indicate that waves are directed away from the event horizon. Dispersion is found to be normal and anomalous at random points in Figures **1** and **2** while it is normal in most of the region in Figure **3**.

For the rotating magnetized plasma, Figures **4** and **6** indicate that waves are directed towards the event horizon while waves move away from the event horizon in Figures **5**, **7** and **8**. Dispersion is normal in Figure **4** while it is anomalous in Figure **5**. Dispersion is normal as well as anomalous randomly in Figures **6**, **7** and **8**. The value of refractive index is less than 1 and also phase and group velocities are antiparallel in all the figures which are the significant features of this unusual medium. Thus, the presence of Veselago medium is confirmed for both rotating (non-magnetized and magnetized) plasmas.

The comparison of the results for isothermal [33] and hot plasma can be summarized in the following table.

Table IV. Comparison of the results

Results	Isothermal Plasma	Hot Plasma
Existence of waves	No waves in rotating magnetized plasma	Waves exist in rotating magnetized plasma
Direction of waves	Some waves move away from the event horizon	Most of the waves move away from the event horizon
Dispersion	Normal at random points	Normal in most of the region in Figures 3 and 4

The difference between our work and previous work is that we have used the variable specific enthalpy and previous work has been done using constant specific enthalpy. In our work, waves exist in rotating magnetized plasma while in previous work, there does not exist any wave in rotating magnetized plasma. Also, we have found that most of the waves move away from the event horizon while for isothermal case, some waves move away from the event horizon. This comparison shows that variation in specific enthalpy effects the direction of waves. Dispersion is normal at random points for isothermal while for hot plasma it is normal in most of the region in Figures **3** and **4**. It is interesting to mention here that the properties of a Veselago medium turn

out for both rotating (non-magnetized and magnetized) plasmas confirming its validity.

Appendix A

This Appendix contains the Maxwell equations, the GRMHD equations for the general line element and the Schwarzschild planar analogue in a Veselago medium ($\epsilon < 0$, $\mu < 0$). In this medium, the Maxwell equations are

$$\nabla.\mathbf{B} = 0, \tag{A1}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{A2}$$

$$\nabla \cdot \mathbf{E} = -\frac{\rho_e}{\epsilon}, \tag{A3}$$

$$\nabla \times \mathbf{B} = -\mu\mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} = 0. \tag{A4}$$

Also, the GRMHD equations take the form

$$\frac{d\mathbf{B}}{d\tau} + \frac{1}{\alpha}(\mathbf{B}.\nabla)\beta + \theta\mathbf{B} = -\frac{1}{\alpha}\nabla \times (\alpha\mathbf{V} \times \mathbf{B}), \tag{A5}$$

$$\nabla.\mathbf{B} = 0, \tag{A6}$$

$$\frac{D\rho_0}{D\tau} + \rho_0\gamma^2\mathbf{V}.\frac{D\mathbf{V}}{D\tau} + \frac{\rho_0}{\alpha}\left\{\frac{g_{,t}}{2g} + \nabla.(\alpha\mathbf{V} - \beta)\right\} = 0, \tag{A7}$$

$$\left\{\left(\rho_0\mu\gamma^2 + \frac{\mathbf{B}^2}{4\pi}\right)\gamma_{ij} + \rho_0\mu\gamma^4 V_i V_j - \frac{1}{4\pi}B_i B_j\right\}\frac{DV^j}{D\tau}$$

$$+\rho_0\gamma^2 V_i\frac{D\mu}{D\tau} - \left(\frac{\mathbf{B}^2}{4\pi}\gamma_{ij} - \frac{1}{4\pi}B_i B_j\right)V^j_{|k}V^k = -\rho_0\gamma^2\mu\{a_i$$

$$-\frac{1}{\alpha}\beta_{j|i}V^j - (\pounds_t\gamma_{ij})V^j\} - p_{|i} + \frac{1}{4\pi}(\mathbf{V} \times \mathbf{B})_i\nabla.(\mathbf{V} \times \mathbf{B})$$

$$-\frac{1}{8\pi\alpha^2}(\alpha\mathbf{B})^2_{|i} + \frac{1}{4\pi\alpha}(\alpha B_i)_{|j}B^j - \frac{1}{4\pi\alpha}(\mathbf{B} \times \{\mathbf{V} \times [\nabla$$

$$\times(\alpha\mathbf{V} \times \mathbf{B}) - (\mathbf{B}.\nabla)\beta] + (\mathbf{V} \times \mathbf{B}).\nabla\beta\})_i, \tag{A8}$$

$$\frac{D}{D\tau}(\mu\rho_0\gamma^2) - \frac{dp}{d\tau} + \Theta(\mu\rho_0\gamma^2 - p) + \frac{1}{2\alpha}(\mu\rho_0\gamma^2 V^i V^j$$

$$+p\gamma_{ij})\pounds_t\gamma_{ij} + 2\mu\rho_0\gamma^2(\mathbf{V}.\mathbf{a}) + \mu\rho_0\gamma^2(\nabla.\mathbf{V}) - \frac{1}{\alpha}\beta^{j,i}$$

$$\times(\mu\rho_0\gamma^2 V_i V_j + p\gamma_{ij}) - \frac{1}{4\pi}(\mathbf{V} \times \mathbf{B}).(\mathbf{V} \times \frac{d\mathbf{B}}{d\tau}) - \frac{1}{4\pi}$$

$$\times(\mathbf{V} \times \mathbf{B}).(\mathbf{B} \times \frac{d\mathbf{V}}{d\tau}) - \frac{1}{4\pi\alpha}(\mathbf{V} \times \mathbf{B}).\nabla\beta - \frac{1}{4\pi}\theta(\mathbf{V} \times \mathbf{B})$$

$$.(\mathbf{V} \times \mathbf{B}) + \frac{1}{4\pi\alpha}(\mathbf{V} \times \mathbf{B}).(\nabla \times \alpha\mathbf{B}) = 0. \tag{A9}$$

For the Schwarzschild planar analogue, β, θ and $\pounds_t\gamma_{ij}$ vanish, the perfect GRMHD equations become

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla \times (\alpha\mathbf{V} \times \mathbf{B}), \tag{A10}$$

$$\nabla.\mathbf{B} = 0, \tag{A11}$$

$$\frac{\partial\rho_0}{\partial t} + (\alpha\mathbf{V}.\nabla)\rho_0 + \rho_0\gamma^2\mathbf{V}.\frac{\partial\mathbf{V}}{\partial t} + \rho_0\gamma^2\mathbf{V}.(\alpha\mathbf{V}.\nabla)\mathbf{V}$$

$$+\rho_0\nabla.(\alpha\mathbf{V}) = 0, \tag{A12}$$

$$\{(\rho_0\mu\gamma^2 + \frac{\mathbf{B}^2}{4\pi})\delta_{ij} + \rho_0\mu\gamma^4 V_i V_j - \frac{1}{4\pi}B_i B_j\}(\frac{1}{\alpha}\frac{\partial}{\partial t} + \mathbf{V}.\nabla)V^j$$

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$$-(\frac{\mathbf{B}^2}{4\pi}\delta_{ij} - \frac{1}{4\pi}B_iB_j)V^j,_kV^k + \rho_0\gamma^2V_i\{\frac{1}{\alpha}\frac{\partial\mu}{\partial t} + (\mathbf{V}.\nabla)\mu\}$$

$$= -\rho_0\mu\gamma^2a_i - p_{,i} + \frac{1}{4\pi}(\mathbf{V}\times\mathbf{B})_i\nabla.(\mathbf{V}\times\mathbf{B}) - \frac{1}{8\pi\alpha^2}(\alpha\mathbf{B})^2,_i$$

$$+\frac{1}{4\pi\alpha}(\alpha B_i),_j B^j - \frac{1}{4\pi\alpha}[\mathbf{B}\times\{\mathbf{V}\times(\nabla\times(\alpha\mathbf{V}\times\mathbf{B}))\}]_i, \qquad (A13)$$

$$(\frac{1}{\alpha}\frac{\partial}{\partial t} + \mathbf{V}.\nabla)(\mu\rho_0\gamma^2) - \frac{1}{\alpha}\frac{\partial p}{\partial t} + 2\mu\rho_0\gamma^2(\mathbf{V}.\mathbf{a}) + \mu\rho_0\gamma^2(\nabla.\mathbf{V})$$

$$-\frac{1}{4\pi}(\mathbf{V}\times\mathbf{B}).(\mathbf{V}\times\frac{1}{\alpha}\frac{\partial\mathbf{B}}{\partial t}) - \frac{1}{4\pi}(\mathbf{V}\times\mathbf{B}).(\mathbf{B}\times\frac{1}{\alpha}\frac{\partial\mathbf{V}}{\partial t})$$

$$+\frac{1}{4\pi\alpha}(\mathbf{V}\times\mathbf{B}).(\nabla\times\alpha\mathbf{B}) = 0. \qquad (A14)$$

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